International Journal of Theoretical Physics, Vol. 33, No. 1, 1994

Algebras of Bounded Operators on Nonclassical Orthomodular Spaces

Hans A. Keller[1] and Herminia Ochsenius A.[2]

Received July 22, 1993

A Hermitian space is called orthomodular if the Projection Theorem holds: every orthogonally closed subspace is an orthogonal summand. Besides the familiar real or complex Hilbert spaces there are non-classical infinite dimensional examples constructed over certain non-Archimedeanly valued, complete fields. We study bounded linear operators on such spaces. In particular we construct an operator algebra $\mathscr{A}$ of von Neumann type that contains no orthogonal projections at all. For operators in $\mathscr{A}$ we establish a representation theorem from which we deduce that $\mathscr{A}$ is commutative. We then focus on a subalgebra $\mathscr{H}$ which turns out to be an integral domain with unique maximal ideal. Both analytic and topological characterizations of $\mathscr{H}$ are given.

1. INTRODUCTION

The study of orthomodular spaces has grown out of the classical theory of Hilbert spaces. The starting point is the fundamental property expressed by the following result.

Projection Theorem. Let H be a real or complex Hilbert space and $\mathscr{L}(H)$ the lattice of all linear subspaces of H. Then

(P) for all $U \in \mathscr{L}(H)$: $U = \bar{U} \Rightarrow H = U \oplus U^{\perp}$

where $\bar{U}$ is the topological closure of U.

Now in a Hilbert space a linear subspace U is topologically closed if and only if it is orthogonally closed, so we may restate (P) as

(P′) for all $U \in \mathscr{L}(H)$: $U = U^{\perp\perp} \Rightarrow H = U \oplus U^{\perp}$

[1]Zentralschweizerisches Technikum Luzern, CH-6048 Horw, Switzerland.
[2]Facultad de Matemáticas, Universidad Católica de Chile, Casilla 306, Santiago, Chile.

0020-7748/94/0100-0001$07.00/0

This is a purely algebraic formulation of the Projection Theorem which makes sense for arbitrary vector spaces endowed with a scalar product. We are led to the following generalization.

Let there be given a vector space V over a field K (of any characteristic), an involution $\xi \mapsto \xi^*$ on K, and an anisotropic *-Hermitian form Φ: $V \times V \to K$. The scalar product Φ implements a symmetric orthogonality relation $\perp$ on V,

$$x \perp y \Leftrightarrow \Phi(x, y) = 0$$

To every $S \subseteq V$ we can form the orthogonal space $S^{\perp} := \{x \in V | x \perp s \text{ for all } s \in S\}$ and the biorthogonal $S^{\perp\perp} := (S^{\perp})^{\perp}$.

Definition. (V, Φ) is called an orthomodular space if

$$\text{(P}') \qquad \text{for all } U \in \mathscr{L}(V): \quad U = U^{\perp\perp} \Rightarrow V = U \oplus U^{\perp}$$

It is readily verified that an anisotropic Hermitian space (V, Φ) satisfies (P′) if and only if the associated lattice $\mathscr{L}_{\perp\perp}(V) := \{U \in \mathscr{L}(V) | U = U^{\perp\perp}\}$ of all orthogonally closed linear subspaces satisfies the orthomodular law.

The most central question is: what orthomodular spaces are there? This question is still far from being solved.

Let us first mention that if $\dim V < \infty$, then the property (P′) is simply equivalent to anisotropy of the form and therefore uninteresting. In infinite dimension, however, the Projection Theorem (P′) is a very strong requirement. We have as classical examples the real or complex Hilbert spaces. For a long time these were the only ones and the endeavors of some people (Gross and Keller, 1977; Morash, 1976; Wilbur, 1977) were directed toward proving that there are no other examples, i.e., that in infinite dimension the algebraic condition (P′) is strong enough to characterize Hilbert spaces.

Then, in 1979/80, a new kind of infinite dimensional orthomodular space was discovered (Gross and Künzi, 1985; Keller, 1980). They are all constructed over certain non-Archimedean, complete fields and are endowed with a natural non-Archimedean norm induced by the form. These new examples are now termed "nonclassical" or "exotic" orthomodular spaces. Over the last decade various of their basic properties have been intensively investigated, e.g., the orthogonal group and Clifford algebras (Fässler-Ullmann, 1983), automorphisms of the underlying orthomodular lattice (Gross, 1987) or measures (Keller, 1990). In the present paper we deal with bounded linear operators. Our purpose is to provide a review of results which illustrate some of the most salient new features that appear in the nonclassical framework.

We shall first outline the construction of an "exotic" orthomodular space V (in Section 2) and then look at operator algebras of von Neumann type. The most straightforward way to define such algebras is by selecting a bounded, self-adjoint operator A and taking $\{A\}'$, the algebra of all operators that commute with A. Accordingly, in Section 4 we will construct a particular computationally suitable operator A on V and examine its basic properties. The algebra $\mathscr{A} = \{A\}'$, which will be studied in Section 5, turns out to be a von Neumann algebra that contains no nontrivial projection; moreover, $\mathscr{A}$ is commutative and without divisors of zero. In Section 6 we focus attention on a subalgebra $\mathscr{H}$ which exhibits remarkable algebraic properties; for example, $\mathscr{H}$ is a local ring and every operator in $\mathscr{H}$ has its spectrum reduced to one point.

We should like to point out that in the present nonclassical setting the powerful tool of spectral decomposition is missing. Nevertheless, our algebra $\mathscr{A}$ contains a countable family of quite simple operators from which all others can be obtained by limiting processes. Such representations can be used to interrelate algebraic and topological properties in our algebras $\mathscr{A}$ and $\mathscr{H}$.

The paper is expository and proofs will be omitted. For details we refer to Gross and Künzi (1985) and Keller and Ochsenius (n.d.).

2. CONSTRUCTION OF A NONCLASSICAL ORTHOMODULAR SPACE

We always let $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$.

2.1. The Base Field

We construct recursively fields F_n ($n \in \mathbb{N}_0$) by

- $F_0 := \mathbb{R}$ ($=$the field of real numbers)
- $F_{n+1} := F_n(X_{n+1})$ = the field of all rational functions in the variable X_{n+1} over F_n

Thus in each step a new transcendental element is adjoined. We order F_{n+1} by powers of X_{n+1}. That means a polynomial

$$p(X_{n+1}) = \alpha_0 + \alpha_1 X_{n+1} + \cdots + \alpha_s X_{n+1}^s \qquad (\alpha_0, \alpha_1, \ldots, \alpha_s \in F_n;\ \alpha_s \neq 0)$$

is positive in F_{n+1} if and only if $\alpha_s > 0$ in F_n. The union $F_\infty := \bigcup_{n=0}^{\infty} F_n$ is naturally an ordered field. Finally we pass to the completion $K := \tilde{F}_\infty$ in the order topology.

Observe that if $\alpha \in F_n = \mathbb{R}(X_1, X_2, \ldots, X_n)$ then

$$m \cdot \alpha < X_{n+1} \qquad \text{for all} \quad m \in \mathbb{N}$$

We say that α is infinitely small with respect to X_{n+1} and we write $\alpha \ll X_{n+1}$. Thus K is a non-Archimedeanly ordered, complete field in which

$$1 =: X_0 \ll X_1 \ll X_2 \ll \cdots \ll X_n \ll \cdots$$

2.2. The Space

The set

$$V := \left\{ (\xi_i)_{i \in \mathbb{N}_0} \in K^{\mathbb{N}_0} \middle| \text{the series } \sum_{i=0}^{\infty} \xi_i^2 X_i \text{ converges in the order topology} \right\}$$

is a vector space over K under componentwise operations. We define a symmetric, bilinear form Φ on V by

$$\Phi(x, y) := \sum_{i=0}^{\infty} \xi_i \eta_i X_i \qquad \text{for} \quad x = (\xi_i)_{i \in \mathbb{N}_0}, \quad y = (\eta_i)_{i \in \mathbb{N}_0} \in V$$

This completes the construction of the quadratic space (V, Φ).

2.3. Basic Properties of (V, Φ)

The most important property is:

Theorem 1. (V, Φ) is an orthomodular space.

Next we observe that the form Φ is positive-definite. Therefore the assignment

$$x \mapsto \|x\| := [\Phi(x, x)]^{1/2} \in \bar{K} \ (= \text{the real closure of } K)$$

is a norm on the vector space V. The norm topology, defined by taking all sets $\{x \in V \mid \|x\| < \varepsilon\}$ (where $\varepsilon \in \bar{K}, \varepsilon > 0$) as a zero-neighborhood basis, turns V into a topological vector space.

Theorem 2. (i) V is complete in the norm topology, i.e., a Banach space.

(ii) A linear subspace U of V is closed in the norm topology if and only if it is orthogonally closed.

Theorems 2 and 3 stress the close analogy to classical Hilbert spaces. However, there are some striking differences of geometric nature, as is illustrated by the following result.

Theorem 3. (i) If two vectors $x, y \in V$ are orthogonal, $x \perp y$, then one of the lengths $\|x\|$, $\|y\|$ is infinitely small with respect to the other.

(ii) V cannot be isometric to a proper subspace.

2.3. The Standard Basis

For every $i \in \mathbb{N}_0$ we let

$$e_i := (0, \ldots, 0, 0, 1, 0, \ldots) \in V$$

be the vector that has 1 in place $i + 1$ and 0 in all other places. Then $e_i \perp e_j$ for $i \neq j$ and $\{e_i | i \in \mathbb{N}_0\}$ is an orthogonal continuous base, which means that every vector $x \in V$ can be expressed as

$$x = \sum_{i=0}^{\infty} \xi_i e_i = \lim_{n \to \infty} \left(\sum_{i=0}^{n} \xi_i e_i \right)$$

However, notice that in view of Theorem 3(i), the base cannot be normalized.

Remark. The above construction can easily be formulated in terms of valuations and can be varied in many ways. For example, we might modify the definition of the form Φ in such a way that the standard base contains, for every $i \in \mathbb{N}_0$, finitely many vectors of square norm X_i (but never infinitely many). Or, we might begin with the function field $F_\infty = \mathbb{R}(X_1, \ldots, X_n, \ldots)$ as before, then endow F_∞ with the valuation v corresponding to the ordering, and then take as ground field the *maximal* completion of (F_∞, v) which is a Henselian field of generalized power series. We do not enter into details.

3. BOUNDED LINEAR OPERATORS

A linear operator B: $V \to V$ is bounded if the set

$$\left\{ \frac{\|B(x)\|}{\|x\|} \middle| 0 \neq x \in V \right\} \subset \bar{K}$$

has an upper bound in $\bar{K}$. Under the usual addition and composition the bounded linear operators on V form an algebra $\mathscr{B}(V)$.

Remark 1. A bounded operator B is certainly continuous in the norm topology but, as shown in Fässler-Ullmann (1983), continuous operators need not be bounded.

Remark 2. In general, a bounded linear operator cannot be assigned a norm in the usual way because norms are in the field $\bar{K}$ where a bounded subset may fail to have a supremum. Nevertheless, there is a natural norm topology on the algebra $\mathscr{B}(V)$ defined by taking the sets

$$\mathscr{U}_\varepsilon := \{B \in \mathscr{B}(V) | \varepsilon \text{ is a bound for } B\} \qquad (\varepsilon \in \bar{K}, \varepsilon > 0)$$

as a zero-neighborhood basis. It is readily verified that $\mathscr{B}(V)$ is complete in this norm topology.

4. THE OPERATOR A

A bounded linear operator $B: V \to V$ is determined by the images $B(e_i)$ of the vectors e_i of the standard basis, hence B can be represented by a countably infinite matrix.

Let

$$u := \sum_{i=0}^{\infty} \frac{1}{X_i} \cdot e_i \in V$$

and consider the operator $A: V \to V$ defined by

$$A(e_i) := u + \left(1 - \frac{1}{X_i}\right) \cdot e_i$$

The matrix of A with respect to the standard basis $\{e_i | i \in \mathbb{N}_0\}$ is

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ \frac{1}{X_1} & 1 & \frac{1}{X_1} & \frac{1}{X_1} & \cdots \\ \frac{1}{X_2} & \frac{1}{X_2} & 1 & \frac{1}{X_2} & \cdots \\ \frac{1}{X_3} & \frac{1}{X_3} & \frac{1}{X_3} & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

The operator A is self-adjoint, $\Phi(A(x), y) = \Phi(x, A(y))$ for all $x, y \in V$.

We first look at the spectrum of A, which is defined as usual by

$$\text{spec}(A) := \{\lambda \in K | (A - \lambda I) \text{ has no inverse in the algebra } \mathscr{B}(V)\}$$

where I is the identity operator.

Theorem 4. $\text{spec}(A) = \{1\}$.

The obvious question is now whether the single point 1 in the spectrum is an eigenvalue of the self-adjoint operator A. The answer is given by the following result:

Theorem 5. The operator A has no eigenvectors at all.

Remark. The fact that A fails to have eigenvalues is not particularly surprising. For, finding eigenvalues of an operator amounts to solving certain infinitary equations over the base field. The point is that since our field K is far from being algebraically closed, its arithmetic structure is most rigid and intricate.

A much stronger result than Theorem 5 can be proved, namely:

Theorem 6. The operator A admits no nontrivial closed invariant subspace. In other words, A does not commute with any nontrivial orthogonal projection.

Theorems 5 and 6 put into evidence that there is little hope for a meaningful spectral theory. The tool of spectral decomposition, so powerful in Hilbert space theory, seems to fade out in the present nonclassical framework.

An easy consequence of Theorem 6 is the following rather peculiar result.

Theorem 7. Every vector $x \neq 0$ is a topologically cyclic vector for the operator A, that is, the linear subspace spanned by x, $A(x)$, $A^2(x), \ldots, A^m(x), \ldots$, is topologically dense in V.

5. THE ALGEBRA $\mathscr{A}$

We consider the commutator algebra

$$\mathscr{A} := \{B \in \mathscr{B}(V) | B \circ A = A \circ B\} \subseteq \mathscr{B}(V)$$

This is an operator algebra of von Neumann type in the sense that (i) $\mathscr{A}$ is closed under taking adjoints, (ii) $\mathscr{A}$ coincides with its double centralizer $\mathscr{A}''$. It follows from Theorem 6 and $\mathscr{A}$ does not contain any nontrivial projection.

From Theorem 7 we easily deduce:

Lemma 8. Let $0 \neq x \in V$. Then every operator B in $\mathscr{A}$ is completely determined by the image vector $B(x)$, that is,

$$\text{for all } B, C \in \mathscr{A}: \quad B(x) = C(x) \Rightarrow B = C$$

This lemma suggests we try to describe the operators in $\mathscr{A}$ by their action on some fixed nonzero vector x. To simplify computations, it is convenient to take $x = e_0$. Thus we introduce the linear injection

$$\Psi: \quad \mathscr{A} \to V \quad \text{defined by} \quad B \mapsto \Psi(B) := B(e_0)$$

We first determine the image $W := \mathrm{Im}(\Psi)$. Recall that V consists of all vectors $x = \sum_{i=0}^{\infty} \xi_i e_i$ subject to the condition that $\sum_{i=0}^{\infty} \xi_i^2 X_i$ converges. Since K is non-Archimedean and complete, the series converges if and only if $\xi_i^2 X_i \to 0$ for $i \to \infty$. Write $\xi_i = \lambda_i / X_i$; then the elements of V are precisely the vectors $x = \sum_{i=0}^{\infty} (\lambda_i / X_i) e_i$ for which $\lambda_i^2 / X_i \to 0$ when $i \to \infty$. Observe that since $X_i \to \infty$, that condition does surely not impose a bound on the numerators λ_i.

Lemma 9. The image W of Ψ consists of all vectors $x = \sum_{i=0}^{\infty} (\lambda_i/X_i)e_i \in V$ for which the set $\{|\lambda_i| \, | i = 0, 1, 2, \ldots\} \subset K$ is bounded.

Remark. We observed previously that there is no meaningful spectral theory at disposition. But we may ask whether it is possible to retain a basic idea of spectral decompositions, namely the idea of building up complicated operators from simpler ones. The above two Lemmas 8 and 9 seem to be promising for that purpose. Indeed, under the correspondence Ψ a typical operator $B \in \mathscr{A}$ is described by the vector $w = B(e_0) \in W$, which is in general an infinite linear combination, $w = \sum_{i=0}^{\infty} \omega_i e_i = \lim_{n\to\infty} w_n$, where $w_n := \sum_{i=0}^{n} \omega_i e_i$. Since all the base vectors e_i are all in W (by Lemma 9), we may first define $C_i := \Psi^{-1}(e_i)$ and then, for all $n \in \mathbb{N}_0$, $B_n := \sum_{i=0}^{n} \omega_i C_i$. Clearly $\Psi(B_n) = \sum_{i=0}^{n} \omega_i e_i = w_n$ and the situation can be pictured as follows:

$$\begin{array}{ccc} w_n = \sum\limits_{i=0}^{n} \omega_i e_i & \xrightarrow{\text{norm topology on } V} & w \\ \Psi \big\uparrow & & \big\uparrow \Psi \\ B_n = \sum\limits_{i=0}^{n} \omega_i C_i & \dashrightarrow & B \end{array}$$

We expect that the operators B_n converge to B in some sense, thereby completing the diagram. In fact, they do so, as we shall prove in Theorem 11 below. Consequently an arbitrary operator B in $\mathscr{A}$ can be represented as an infinite sum $B = \sum_{i=0}^{\infty} \omega_i C_i$. The crucial point is to find the right topology.

We first show that the operators C_i corresponding to the base vectors e_i can be explicitly computed.

Theorem 10. For $i \in \mathbb{N}_0$ put $\rho_i := 1 - 1/X_i$ and let

$$C_i := I + \rho_i(A - \rho_i I)^{-1}$$

where I is the identity. Then $C_i(e_0) = e_i$, that is, $C_i = \Psi^{-1}(e_i)$.

Remark. The inverse $(A - \rho_i I)^{-1}$ exists because $\rho_i \notin \operatorname{spec}(A) = \{1\}$. In fact it is not hard to compute the matrix of $\{A - \rho_i I)^{-1}$ and hence also of C_i.

Theorem 11. Let $B \in \mathscr{A}$ and write $B(e_0) = w = \sum_{i=0}^{\infty} \omega_i e_i$. Then the operators $B_n := \sum_{i=0}^{n} \omega_i C_i \in \mathscr{A}$ converge to B in the topology of pointwise convergence. Hence B can be represented as

$$B = \lim_{n\to\infty} \left(\sum_{i=0}^{n} \omega_i C_i \right) = \sum_{i=0}^{\infty} \omega_i C_i.$$

Clearly $\mathscr{A}$ is closed in the algebra $\mathscr{B}(V)$ with respect to the topology of pointwise convergence, so we deduce:

Corollary 12. Let $\mathscr{C}$ be the subalgebra of $\mathscr{B}(V)$ generated algebraically by $\{C_i | i \in \mathbb{N}_0\}$. Then $\mathscr{A}$ equals the closure of $\mathscr{C}$ with respect to the topology of pointwise convergence.

From their definition it is evident that the operators C_i commute with each other, so we have the following rather unexpected result.

Corollary 13. The algebra $\mathscr{A}$ is commutative.

6. THE SUBALGEBRA $\mathscr{H}$

In the above section we have seen that $\mathscr{A}$ is a commutative algebra without divisors of zero, i.e., an integral domain. It is natural to study $\mathscr{A}$ along the lines of well-established concepts of commutative algebra. In other words, one should examine questions such as the existence of prime elements, unique factorization, prime ideals, maximal ideals, and so on. When attacking such problems we realized that the algebra $\mathscr{A}$ is extremely complicated as ring-theoretic aspects are concerned. The reason is that $\mathscr{A}$ contains a huge variety of operators of utmost dissimilar characteristics. However, we then discovered that by imposing an analytic condition it is possible to single out a subalgebra $\mathscr{H}$ which exhibits nice algebraic features and which contains all operators that are important in applications.

The definition of $\mathscr{H}$ relies on the correspondence Ψ: $\mathscr{A} \to W$.

Definition. Let

$$U := \left\{ w = \sum_{i=0}^{\infty} \frac{\lambda_i}{X_i} e_i \in W \,\middle|\, \text{the sequence } (\lambda_i)_{i \in \mathbb{N}_0} \text{ is convergent} \right\}$$

$$\mathscr{H} := \{ B \in \mathscr{A} \,|\, B(e_0) \in U \}$$

It is obvious that U is a linear subspace of W, so $\mathscr{H}$ is closed under sums and scalar multiples. A somewhat lengthy computation establishes that $\mathscr{H}$ is also closed under products, hence $\mathscr{H}$ is a subalgebra. It is easily checked that the operators A, I, and all C_i belong to $\mathscr{H}$.

For the operators in $\mathscr{H}$ there is an efficient criterion on invertibility. We first mention that if $B \in \mathscr{H}$ has an inverse in $\mathscr{A}$, then this inverse is in $\mathscr{H}$. Thus invertibility in $\mathscr{A}$ is the same as invertibility in $\mathscr{H}$. Now consider any $B \in \mathscr{H}$ and write $B(e_0) = \sum_{i=0}^{\infty} (\lambda_i / X_i) e_i$. Here the sequence $(\lambda_i)_{i \in \mathbb{N}_0}$ is convergent, so we can attach to B the quantities

$$\lambda_B := \lim_{i \to \infty} \lambda_i \qquad \text{and} \qquad \delta_B := \sum_{i=0}^{\infty} (\lambda - \lambda_i)$$

We then have:

Theorem 14. The operator $B \in \mathscr{H}$ is invertible if and only if $\lambda_B \neq \delta_B$.

Corollary 15. Let $B \in \mathscr{H}$ and let $D := B - \mu \cdot I$ for some $\mu \in K$. Then D is noninvertible if and only if $\mu = \lambda_B - \delta_B$.

Corollary 16. Every operator in $\mathscr{H}$ has a one-point spectrum.

The spectrum of A is $\{1\}$, so $A - I$ is not invertible and the same holds for all elements in $(A - I)\mathscr{H}$, the principal ideal generated by $A - I$. It turns out that $A - I$ can be considered as a prototype of noninvertibles in $\mathscr{H}$.

Theorem 17. An operator $B \in \mathscr{H}$ is noninvertible if and only if it belongs to $\mathscr{M} := \overline{(A - I)\mathscr{H}}$, the closure of $(A - I)\mathscr{H}$ in the norm topology on $\mathscr{B}(V)$. Hence $\mathscr{H}$ is a local ring and $\mathscr{M}$ is its unique maximal ideal.

To conclude, we remark that the subalgebra $\mathscr{H}$ can be characterized differently. The way $\mathscr{H}$ was introduced above is highly efficient for techniques of proofs, but is unsatisfactory inasmuch as it depends on the continuous basis $\{e_i\}$ and on the choice of the referential vector e_0 in the representation Ψ: $\mathscr{A} \to W$. One can establish the following:

Theorem 18. Let $\mathscr{C}'$ be the subalgebra of $\mathscr{A}$ generated by $\{A\} \cup \{C_i | i \in \mathbb{N}_0\}$. Then $\mathscr{H}$ equals the closure of $\mathscr{C}'$ in the norm topology on $\mathscr{B}(V)$.

This result nicely fits into the leading ideas of the present investigation, which might be summarized as follows. In our nonclassical framework the base field is not algebraically closed. As a consequence, the significance of spectra is strongly diminished, since it is no longer possible to reconstruct a (self-adjoint) operator from its spectral family of orthogonal projections. We believe that there are other ways of representing operators by suitable families of "elementary" operators and that such representations provide insight into essential features of the operator algebras under consideration. For our algebras $\mathscr{A}$ and $\mathscr{H}$ this is indeed the case. The elementary operators C_i have first been introduced by means of the correspondence Ψ as the matches of the base vectors e_i. We then showed that they can be defined independently and computed by $C_i = I + \rho_i(A - \rho_i I)^{-1}$. Here $\rho_i = 1 - 1/X_i$, so the sequence $(\rho_i)_{i \in \mathbb{N}_0}$ tends to 1, i.e., to the single point in the spectrum of A. The family $\{C_i | i \in \mathbb{N}_0\}$ contains basic information on the operator A and the von Neumann algebra $\mathscr{A}$ generated by $\{A\}$. In fact, in Section 5 we have seen that every operator B commuting with A can be obtained from the C_i by forming finite linear combinations and then taking limits in the topology of pointwise convergence. This representation entails that $\mathscr{A}$ is commutative. Now it turns out that taking limits in the norm topology yields the algebraically distinguished subalgebra $\mathscr{H}$.

ACKNOWLEDGMENT

This work was supported in part by the Dirección de Investigación de la Universidad Católica DIUC Proyecto 91/033.

REFERENCES

Fässler-Ullmann, A. (1983). On non-classical Hilbert spaces, *Expositiones Matematicae*, **3**, 275–277.

Gross, H. (1987). Different orthomodular complementations on a lattice, *Order*, **4**, 79–92.

Gross, H., and Keller, H. A. (1977). On the definition of Hilbert space, *Manuscripta Mathematica*, **23**, 67–90.

Gross, H., and Künzi, U. M. (1985). On a class of orthomodular quadratic spaces, *L'Enseignement Mathématique*, **31**, 187–212.

Keller, H. A. (1980). Ein nicht-klassischer Hilbertscher Raum, *Mathematische Zeitschrift*, **172**, 41–49.

Keller, H. A. (1990). Measures on infinite-dimensional orthomodular spaces, *Foundations of Physics*, **20**, 575–604.

Keller, H. A., and Ochsenius, H. (n.d.). Algebras of operators on non-classical orthomodular spaces, in preparation.

Morash, R. P. (1976). Orthomodularity and non-standard constructions, *Glasnik Matematicki*, **10**, 231–239.

Wilbur, W. J. (1977). On characterizing the standard quantum logics, *Transactions of the American Mathematical Society*, **233**, 265–282.